JOURNAL OF COMPUTATIONAL PHYSICS **51**, 527–535 (1983)

Note

Periodic Solutions of Differential Equations and One-Parameter Family of Operators

A method for computing bounds to periodic solutions of the system

$$y' = f(t, y), \qquad y, f \in R^n, \quad f \text{ periodic in } t \text{ with period } T, \tag{1}$$

is studied using an integral equation

$$y = G_\lambda y, \tag{2}$$

depending on a parameter λ, and whose solutions for each λ are the periodic solutions of (1). λ is then given a value λ_0 so as to determine a set S for which $G_{\lambda_0} S \subset S$. Since G_{λ_0} is proved to satisfy Schauder's conditions, then in S there is a solution of (2), i.e., a periodic solution of (1). The method, which evidently also constitutes an existence proof in S, has the peculiarity of being quite general since it can be used to bound other kinds of solutions of the system $y' = f(t, y)$, by simply setting up a different integral equation $y = G_\lambda y$ which possesses the solutions to be bounded. This method has then been used to bound periodic solutions of a system arising from the dynamics of two floating bodies, for which $G_\lambda S \subset S$ has the form of a system of inequalities.

1. Introduction

Let $f(t, y)$ be a continuous function $f: R \times R^n \to R^n$, periodic in t with period T. For the norm of $y = (y_1, y_2, ..., y_n)$ in R^n we take $\sum_{i=1}^n |y_i|$ and use the symbol $|y|$ instead of $\|y\|$ in order to avoid confusion when defining later the norm of y as an n-valued function on $[0, T]$. Let $C[0, T]$ be the vector space of continuous functions $y: [0, T] \to R^n$, with the norm $\|y\| = \max_{t \in [0,T]} |y|$, and let D be a set in $C[0, T]$. The problem considered here is that of determining bounds for periodic solutions in D of the system

$$y' = f(t, y), \qquad 0 \leqslant t, \ |y| < \infty. \tag{1}$$

The following method reduces the problem to that of finding a solution $\lambda \in R^n$ of an inclusion relation (often defined as a system of inequalities). At first, a function $\Psi(\lambda, t, y)$ depending on a parameter $\lambda \in \Lambda$ is set up so that the solutions of the integral equation

$$A(\lambda, t) y = \int_0^t \Psi(\lambda, \tau, y)\, d\tau + c, \qquad c \in R^n, \tag{2}$$

where A is a nonsingular matrix, depend on c but not on λ (i.e., invariant with λ). A simple example of invariance of solutions for equations in one variable is given by

0021-9991/83 $3.00

$F(x)+\lambda g(x)=0$, $\lambda \geqslant 0$, where F and g are nonnegative functions having the same zeros. Besides, Ψ must have the property that (2) is equivalent to (1), i.e., all the solutions of (1) are also solutions of (2) and vice versa. Then we determine c so that the corresponding integral equation, which we write briefly $y=G_\lambda y$ (it always contains λ), has, as solutions, only the T-periodic solutions of (1). λ is at this point a free parameter which is used to find bounds for the solutions of $y=G_\lambda y$, i.e., T-periodic solutions of (1). In fact, assuming that for a given $\lambda=\lambda_0$ we can determine a bounded set $S \subset C[0, T]$, $S \subset D$, such that $G_{\lambda_0} S \subset S$, then in S there is a T-periodic solution of (1) since G_{λ_0} satisfies the conditions of Schauder's theorem (Theorems 3 and 4). In some cases of applied physics (see Section 3) $G_\lambda S \subset S$ is a system of inequalities and λ_0 is a solution of such a set. The above method of bounding the solutions which evidently has the property of being at the same time an existence theorem, allows the use of iterative schemes when not only bounds but also numerical values of the solution are sought. In fact the use of iterative schemes, such as the generalization of the Picard method for initial value problems, see [2], requires, as a main condition, that all iterations are bounded, which will be the case if $G_{\lambda_0} S \subset S$ and the starting point is in S. In Section 2 we prove the equivalence theorem for (2) and derive the equation $y=G_\lambda y$. We also show that G_λ satisfies Schauder's conditions for every $\lambda \in \Lambda$. In Section 3 we use the results of Section 2 to compute bounds for periodic solutions of a second-order damped system with forcing term. This is done simply by finding some λ solution of a system of inequalities.

2. The Class G_λ

Let $A(\lambda, t)$ be a nonsingular matrix whose elements are functions of λ and t, $\lambda \in R^n$, $0 \leqslant t < \infty$, and let dA/dt be the derivative of the matrix A with respect to t. Assume that dA/dt has elements continuous with respect to t, and let

$$\Psi = \frac{dA}{dt}\, y + Af(t, y),$$

where f is the function occurring in (1).

Theorem 1. *For any fixed λ the integral equation*

$$A(\lambda, t)y = \int_0^t \Psi(\lambda, \tau, y)\, d\tau + c, \qquad c \in R^n, \tag{3}$$

is equivalent to (1).

Proof. Let y^* be any solution of (1). We find from (3)

$$\frac{d}{dt}\left\{A(\lambda, t)y^* - \int_0^t \Psi(\lambda, \tau, y^*)\, d\tau\right\} = 0,$$

i.e.,

$$Ay^* - \int_0^t \Psi \, d\tau = \text{constant}.$$

Conversely, if y^* is a continuous solution of (3) we have

$$\frac{dA}{dt} y^* + A(\lambda, t) \frac{dy^*}{dt} = \frac{dA}{dt} y^* + A(\lambda, t) f(t, y^*),$$

and the theorem is proved.

As an example let (1) be a system of the form

$$y' = S(t)y + F(t, y), \qquad F, y \in R^n.$$

with S an $n \times n$ matrix. If Y is a fundamental matrix of $y' = Sy$, i.e., one whose columns are linearly independent solutions of $y' = Sy$, we may take $A(\lambda, t) = Y^{-1}(t)$ and with some manipulations we get for (3)

$$y = \int_0^t Y(t)\, Y^{-1}(\tau)\, F(\tau, y)\, d\tau + Y(t)c,$$

which is the classical transformation of the foregoing system into a system of integral equations.

If we restrict ourselves to T-periodic solutions of (1), i.e., solutions of the system

$$\begin{aligned} y' &= f(t, y), \qquad 0 \leqslant t \leqslant T, \\ y(0) &= y(T), \end{aligned} \tag{4}$$

and assume that $A(\lambda, 0) \neq A(\lambda, T)$, we get from (3) and (4)

$$y = A^{-1}(\lambda, t) \int_0^t \Psi(\lambda, \tau, y)\, d\tau + A^{-1}(\lambda, t)\, B^{-1} \int_0^T \Psi(\lambda, \tau, y)\, d\tau, \tag{5}$$

where

$$B = A(\lambda, T)\, A^{-1}(\lambda, 0) - I.$$

The right-hand side of (5) defines an operator $G_\lambda : C[0, T] \to C[0, T]$ (depending on λ) and it is easy to see that (5), which may be written as

$$y = G_\lambda y,$$

has the same continuous solutions as (4) for each λ.

Before showing (Theorems 3 and 4) that G_λ satisfies the conditions of Schauder's theorem, we recall the definition of relative compactness and the statement of Schauder's theorem, see [1, p. 456].

DEFINITION. A subset S of a metric space is called relatively compact if every sequence in S contains a convergent subsequence (i.e., if $\bar{S}$ is compact).

THEOREM 2 (Schauder). *Let S be a convex closed subset of a normed space. Let H be a continuous mapping of S into a relatively compact subset of S. Then H has a fixed point in S.*

THEOREM 3. *Let G_λ be the operator defined by (5), Λ a set in R^n, and let $S \subset C[0, T]$ be a bounded set such that for any $\lambda \in \Lambda$, $G_\lambda S \subset S$. Then $G_\lambda S$ is relatively compact for any $\lambda \in \Lambda$.*

Proof. Since $\Psi(\lambda, t, x)$ is a continuous vector function of t and x, $0 \leqslant t \leqslant T$, $x \in R^n$, there exists $M > 0$ (depending on λ) such that $|\Psi| < M$ when $x \in S$. From (5) the equation $Ay = AG_\lambda x$ takes the form

$$A(\lambda, t)y = \int_0^t \Psi(\lambda, \tau, x)\, d\tau + B^{-1} \int_0^T \Psi(\lambda, \tau, x)\, d\tau. \tag{6}$$

We have (for any $y \in G_\lambda S$)

$$A(\lambda, t+h)\, y(t+h) - A(\lambda, t)\, y(t) = \int_t^{t+h} \Psi(\lambda, \tau, x)\, d\tau, \qquad t+h \in [0, T],$$

or equivalently

$$A(\lambda, t+h)\{ y(t+h) - y(t)\} + \{A(\lambda, t+h) - A(\lambda, t)\}\, y(t) = \int_t^{t+h} \Psi(\lambda, \tau, x)\, d\tau.$$

Then, for every $\lambda \in \Lambda$,

$$\begin{aligned} |y(t+h) - y(t)| \leqslant & \left| A^{-1}(\lambda, t+h) \frac{dA(\lambda, t+\theta h)}{dt} h y(t) \right| \\ & + \left| A^{-1}(\lambda, t+h) \int_t^{t+h} \Psi(\lambda, \tau, x)\, d\tau \right|, \end{aligned}$$

where $0 < \theta < 1$. Recalling that $|\ |$ is the l_1-norm in R^n, and using for the norm $\|\ \|$ of matrix the corresponding $\|\ \|_1$ norm, we have

$$|y(t+h) - y(t)| \leqslant h(\max_{t \in [0,T]} \|A^{-1}\|) \left(\max_{t \in [0,T]} \left\| \frac{dA}{dt} \right\| N + M \right), \qquad y \in G_\lambda S,$$

where N is a bound on y (from $G_\lambda S \subset S$, S bounded). Hence $G_\lambda S$ is equicontinuous on $[0, T]$, and since for each t in $[0, T]$ the set $\{ y(t)\colon y \in S\}$ is relatively compact in R^n (it is bounded), it follows from a generalization of Ascoli's theorem, see [3, p. 168], that $G_\lambda S$ is relatively compact.

THEOREM 4. *The mapping G_λ defined by* (5) *is continuous on* $C[0,T]$.

Proof. Let $x_0 \in C[0,T]$ and let m be a positive number. In the closed region $[0,T] \times V$, where V is the closed ball $\{x \in R^n : |x - x_0| \leqslant m\}$, Ψ is uniformly continuous (for any fixed λ). Then, given $\sigma_1 > 0$, there is a $\delta \leqslant m$ such that $|\Psi(\lambda,\tau,x) - \Psi(\lambda,\tau,x_0)| < \sigma_1$ whenever $|x - x_0| < \delta$. From (6), taking $x \in C[0,T]$ such that $|x(t) - x_0(t)| < \delta$, we have (if $y_0 = G_\lambda x_0$)

$$|y(t) - y_0(t)| \leqslant \left| A^{-1}(\lambda,t) \int_0^t (\Psi(\lambda,\tau,x) - \Psi(\lambda,\tau,x_0))\, d\tau \right| + \left| A^{-1}(\lambda,t)\, B^{-1} \int_0^T (\Psi(\lambda,\tau,x) - \Psi(\lambda,\tau,x_0))\, d\tau \right|,$$

which, using the maximum norm in $C[0,T]$, becomes

$$\| y - y_0 \| \leqslant \sigma_1 T(1 + \|B^{-1}\|) \max_{t \in [0,T]} \|A^{-1}\|.$$

Given any σ, it is sufficient to take a δ corresponding to $\sigma_1 < \sigma/(T(1 + \|B^{-1}\|) \max_{t\in[0,T]} \|A^{-1}\|)$ in order to get

$$\| y - y_0 \| < \sigma \qquad \text{when} \qquad \| x - x_0 \| < \delta,$$

which proves that G_λ is continuous at x_0.

3. APPLICATION TO A SECOND-ORDER SYSTEM

The problem of the dynamic behaviour of two close floating bodies has arisen at CERN. The corresponding mathematical system is

$$\begin{aligned} u'' + b_1 |u'|\, u' + c_1 u + d(u - v)' &= F \sin(\omega t + \phi), \\ v'' + b_2 |v'|\, v' + c_2 v - d(u - v)' &= \mathrm{E} \sin(\omega t), \end{aligned} \tag{7}$$

where $b_1, c_1, d, b_2, c_2, F, E, \omega, \phi$ are positive coefficients. We shall determine a set of values of the coefficients such that, for each value in the set, there exists a periodic solution of (7) with period $T = 2\pi/\omega$.

Putting $u = y_1$, $u' = y_2$, $v = y_3$, $v' = y_4$, (7) takes the form of (1)

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -c_1 y_1 - d y_2 + d y_4 - b_1 |y_2|\, y_2 + F \sin(\omega t + \phi), \\ y_3' &= y_4, \\ y_4' &= d y_2 - c_2 y_3 - d y_4 - b_2 |y_4|\, y_4 + E \sin(\omega t). \end{aligned} \tag{8}$$

We take for $A(\lambda, t)$ the following matrix, in which for simplicity $\lambda \in R$ (instead of $\lambda \in R^n$), $\lambda > 0$,

$$A(\lambda, t) = \begin{pmatrix} e^{\lambda t} & e^{\lambda t} & 0 & 0 \\ 0 & e^{(\lambda+1)t} & 0 & 0 \\ 0 & 0 & e^{\lambda t} & e^{\lambda t} \\ 0 & 0 & 0 & e^{(\lambda+1)t} \end{pmatrix}.$$

Multiplying (5) by $A(\lambda, t)$ and replacing y by x in the right-hand side we get from (8)

$$\begin{aligned} e^{\lambda t}(y_1 + y_2) &= \int_0^t \Psi_1\, d\tau + \int_0^T \Psi_1\, d\tau/(e^{\lambda T} - 1), \\ e^{(\lambda+1)t} y_2 &= \int_0^t \Psi_2\, d\tau + \int_0^T \Psi_2\, d\tau/(e^{(\lambda+1)T} - 1), \\ e^{\lambda t}(y_3 + y_4) &= \int_0^t \Psi_3\, d\tau + \int_0^T \Psi_3\, d\tau/(e^{\lambda T} - 1), \\ e^{(\lambda+1)t} y_4 &= \int_0^t \Psi_4\, d\tau + \int_0^T \Psi_4\, d\tau/(e^{(\lambda+1)T} - 1), \end{aligned} \tag{9}$$

where

$$\begin{aligned} \Psi_1 &= e^{\lambda\tau}\{(\lambda - c_1)x_1 + (\lambda + 1 - d)x_2 - b_1|x_2|x_2 + dx_4 + F\sin(\omega\tau + \phi)\}, \\ \Psi_2 &= e^{(\lambda+1)\tau}\{-c_1x_1 + (\lambda + 1 - d)x_2 - b_1|x_2|x_2 + dx_4 + F\sin(\omega\tau + \phi)\}, \\ \Psi_3 &= e^{\lambda\tau}\{dx_2 + (\lambda - c_2)x_3 + (\lambda + 1 - d)x_4 - b_2|x_4|x_4 + E\sin(\omega\tau)\}, \\ \Psi_4 &= e^{(\lambda+1)\tau}\{dx_2 - c_2x_3 + (\lambda + 1 - d)x_4 - b_2|x_4|x_4 + E\sin(\omega\tau)\}. \end{aligned}$$

In operator form,

$$Ay = AG_\lambda x, \qquad x, y \in C[0, T].$$

We now determine a closed convex set D_λ such that $G_\lambda D_\lambda \subset D_\lambda$. We define D_λ by the inequalities

$$|x_i| \leqslant M_i, \qquad i = 1,..., 4.$$

and choose M_i together with λ and the coefficients of (7), in such a way that

$$|y_i| \leqslant M_i, \qquad i = 1,..., 4.$$

Assume

$$\lambda > \max\{c_1, c_2, d - 1\}, \tag{10}$$

$$M_2 \leqslant (\lambda + 1 - d)/b_1, \tag{11}$$

$$M_4 \leqslant (\lambda + 1 - d)/b_2. \tag{12}$$

Using $|x_i| \leqslant M_i$ and (10)–(12), we get from (9)

$$|y_1 + y_2| \leqslant \frac{\lambda - c_1}{\lambda} M_1 + \frac{(\lambda + 1 - d)^2}{4b_1\lambda} + \frac{dM_4}{\lambda} + \frac{F}{\lambda},$$

$$|y_2| \leqslant \frac{c_1}{\lambda + 1} M_1 + \frac{(\lambda + 1 - d)^2}{4b_1(\lambda + 1)} + \frac{dM_4}{\lambda + 1} + \frac{F}{\lambda + 1},$$

$$|y_3 + y_4| \leqslant \frac{d}{\lambda} M_2 + \frac{\lambda - c_2}{\lambda} M_3 + \frac{(\lambda + 1 - d)^2}{4b_2\lambda} + \frac{E}{\lambda},$$

$$|y_4| \leqslant \frac{d}{\lambda + 1} M_2 + \frac{(\lambda + 1 - d)^2}{4b_2(\lambda + 1)} + \frac{c_2}{\lambda + 1} M_3 + \frac{E}{\lambda + 1}.$$

Similarly, the condition $|y_i| \leqslant M_i$ will be satisfied if

$$\frac{\lambda - c_1}{\lambda} M_1 + \frac{(\lambda + 1 - d)^2}{4b_1\lambda} + \frac{dM_4}{\lambda} + \frac{F}{\lambda} \leqslant M_1 - M_2,$$

$$\frac{c_1}{\lambda + 1} M_1 + \frac{(\lambda + 1 - d)^2}{4b_1(\lambda + 1)} + \frac{dM_4}{\lambda + 1} + \frac{F}{\lambda + 1} \leqslant M_2,$$

$$\frac{d}{\lambda} M_2 + \frac{\lambda - c_2}{\lambda} M_3 + \frac{(\lambda + 1 - d)^2}{4b_2\lambda} + \frac{E}{\lambda} \leqslant M_3 - M_4,$$

$$\frac{d}{\lambda + 1} M_2 + \frac{(\lambda + 1 - d)^2}{4b_2(\lambda + 1)} + \frac{c_2}{\lambda + 1} M_3 + \frac{E}{\lambda + 1} \leqslant M_4,$$

a system which takes the form

$$\begin{aligned}
&\frac{(\lambda + 1 - d)^2}{4b_1} + dM_4 + F \leqslant c_1 M_1 - \lambda M_2, \\
&\frac{(\lambda + 1 - d)^2}{4b_1} + dM_4 + F \leqslant (\lambda + 1) M_2 - c_1 M_1, \\
&\frac{(\lambda + 1 - d)^2}{4b_2} + dM_2 + E \leqslant c_2 M_3 - \lambda M_4, \\
&\frac{(\lambda + 1 - d)^2}{4b_2} + dM_2 + E \leqslant (\lambda + 1) M_4 - c_2 M_3.
\end{aligned} \tag{13}$$

We recall that to each solution of (10)–(13) corresponds a $D_\lambda : |x_i| \leqslant M_i$ and $G_\lambda D_\lambda : |y_i| \leqslant M_i$ such that $G_\lambda D_\lambda \subset D_\lambda$.

We try to solve the system of inequalities (10)–(13) (where the unknowns are the coefficients in (7), together with λ and M_i) by taking

$$
\begin{aligned}
M_2 &= \frac{\lambda + 1 - d}{b_1}, \\
M_4 &= \frac{\lambda + 1 - d}{b_2}, \\
c_1 M_1 &= \lambda M_2 + \tfrac{1}{2} M_2, \\
c_2 M_3 &= \lambda M_4 + \tfrac{1}{2} M_4,
\end{aligned} \tag{14}
$$

which, substituted in (13), gives

$$
\begin{aligned}
b_1 M_2^2 + 4dM_4 + 4F &\leqslant 2M_2, \\
b_2 M_4^2 + 4dM_2 + 4E &\leqslant 2M_4.
\end{aligned} \tag{15}
$$

The solutions of the system formed by the seven inequalities (10), (14), and (15) are evidently also solutions of (10)–(13) (but not vice versa). Using the following theorem which gives solutions of (15), we obtain a set of solutions of (10), (14), and (15).

THEOREM 5. *A set of solutions of* (15) *is given by*

$$
4F < \frac{1}{b_1}, \qquad 4E < \frac{1}{b_2}, \qquad M_2 = \frac{1}{b_1}, \qquad M_4 = \frac{1}{b_2},
$$

$$
4d \leqslant \min \left\{ b_2 \left(\frac{1}{b_1} - 4F \right), b_1 \left(\frac{1}{b_2} - 4E \right) \right\}.
$$

Proof. Assume $4F < 1/b_1$, $4E < 1/b_2$. It is easy to see that for $M_2 = 1/b_1$, $M_4 = 1/b_2$ the expressions $2M_2 - b_1 M_2^2 - 4F$ and $2M_4 - b_2 M_4^2 - 4E$ take their maximum value at $(1/b_1) - 4F > 0$ and $(1/b_2) - 4E > 0$, respectively. Substituting values taken from the set $4F < 1/b_1$, $4E < 1/b_2$, $M_2 = 1/b_1$, $M_4 = 1/b_2$ into (15) it is evident that the two inequalities are solved if $4dM_4 \leqslant (1/b_1) - 4F$ and $4dM_2 \leqslant (1/b_2) - 4E$.

We then have the following solutions of (10), (14), (15):

$$
\lambda = d, \qquad d \geqslant \max\{c_1, c_2\}, \qquad d \leqslant \min \left\{ \frac{b_2}{4} \left(\frac{1}{b_1} - 4F \right), \frac{b_1}{4} \left(\frac{1}{b_2} - 4E \right) \right\},
$$

$$
M_1 = \left(d + \frac{1}{2} \right) \Big/ c_1 b_1, \qquad M_2 = \frac{1}{b_1}, \qquad M_3 = \left(d + \frac{1}{2} \right) \Big/ c_2 b_2, \qquad M_4 = \frac{1}{b_2},
$$

$$
4F < \frac{1}{b_1}, \qquad 4E < \frac{1}{b_2}.
$$

In conclusion, for values which satisfy the above inequalities and for any ω and ϕ, there exists a $2\pi/\omega$-periodic solution of (7). For example, the following values suggested by physical requirements are found to satisfy our inequalities

$$b_1 = 0.1, \qquad b_2 = 0.1, \qquad c_1 = 0.3, \qquad c_2 = 0.2,$$
$$d = 0.3, \qquad F = 1, \qquad E = 0.5, \qquad \phi = 0.5, \qquad \omega = 2,$$

which yields $\lambda = 0.3$ and the solution bounds $M_1 = 27$, $M_3 = 40$, i.e.,

$$-27. \leqslant u \leqslant 27.,$$
$$-40. \leqslant v \leqslant 40..$$

We have also computed some solutions (one with the above data) using the iterative scheme $y_{n+1} = G_\lambda y_n$. In all cases the iterates converged numerically to curves which were verified to be periodic solutions.

Acknowledgments

I am indebted to the Referee for his valuable comments and penetrating criticism.

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Received: July 27,1982; revised: January 4, 1983

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